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The dynamic renormalisation group in the large-*n* limit

P Szépfalusy† and T Tél‡

† Central Research Institute for Physics, H-1525 Budapest POB 49, Hungary
‡ Institute for Theoretical Physics, Eötvös University, H-1445 Budapest, POB 327, Hungary

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Abstract. An exact formulation of the Wilson-type renormalisation group transformation for the path probability of the time-dependent Ginzburg–Landau model is given when the number of components n of the order parameter goes to infinity. The parameter space containing an infinite number of coupling constants can be specified as Taylor coefficients of a function of ϕ^2 and $\tilde{\phi}\phi$, where ϕ and $\tilde{\phi}$ are the order parameter and its conjugate Martin–Siggia–Rose field respectively. For the fixed point expression of this function an integral equation is derived whose solution is presented for the first few coefficients.

1. Introduction

Although the renormalisation group method has been sucessfully applied in recent years to a wide class of models in order to calculate their dynamic critical properties (see, for a recent review, Hohenberg and Halperin 1977), the general features of the dynamic renormalisation group (DRG) are less well understood than those of the static one. In this respect explicit solutions, which can be achieved when simplifying situations occur, are very useful. This is the case when the number of components n of the order parameter goes to infinity. In particular a most detailed investigation of the static renormalisation group in this limit was carried out by Ma (1973, 1974a, b), during which many important features were illustrated (see furthermore Zanetti and Di Castro 1977).

The purpose of the present paper is to study the corresponding dynamical problem. We examine a model with purely relaxational dynamics, namely the time-dependent Ginzburg–Landau model in the large-n limit, which is non-trivial but simple enough to be tractable analytically. By using the funtional integral formalism we are able to give a full description of the DRG in which the transformation of an infinite number of dynamic parameters is involved. For the fixed point expression of the function specifying the static as well as the dynamic parameters, an integral equation is derived. The solution is presented in the form of a Taylor expansion. Finally the wavenumber and frequency dependences of the couplings are discussed.

2. The model

We start from the 'Hamiltonian' (Ma 1973, 1974a)

$$\mathscr{H} = \int \mathrm{d}^d x \, [a_0(\nabla\phi)^2 + U(\phi^2)], \tag{1}$$

$$\phi^2 \equiv \frac{1}{2}\sum_{j=1}^{n} \phi_j^2, \qquad (\nabla\phi)^2 \equiv \frac{1}{2}\sum_{j=1}^{n} (\nabla\phi_j)^2 \tag{2}$$

with cut-off Λ and dimensionality d. $\phi_j(x, t)$ stands for the n-component real-order parameter, a_0 is a positive constant, and $U(\phi^2)$ is a power series in ϕ^2. The spherical model limit is defined by the condition $n \to \infty$ (Stanley 1968).

The dynamics is based on the equation of motion (Halperin *et al* 1972, 1974)

$$\partial\phi_j/\partial t = -\Gamma_0\, \delta\mathscr{H}/\delta\phi_j + \zeta_j, \qquad j = 1, 2, \ldots, n. \tag{3}$$

Γ_0 is a real constant ($\Gamma_0 > 0$) and $\zeta_j(x, t)$ is a Gaussian white noise, the correlation function of which is proportional to $2\Gamma_0$.

The model can be described equivalently by the path probability $P\{\phi(x, t)\}$ (Onsager and Machlup 1953a, b, Kubo *et al* 1973, Graham 1973) or rather by the action $\mathscr{A}\{\tilde{\phi}, \phi\}$, where $\tilde{\phi}_j(x, t)$ is the conjugate Martin–Siggia–Rose field (Martin *et al* 1973, De Dominicis 1976, Janssen 1976, Bausch *et al* 1976, De Dominicis and Peliti 1978). The connection between these quantities can be written as

$$P\{\phi(x, t)\} \propto \int \prod_{j=1}^{n} \mathrm{d}\tilde{\phi}_j\, \mathrm{e}^{\mathscr{A}\{\tilde{\phi},\phi\}}. \tag{4}$$

The weight functional

$$W\{\tilde{\phi}, \phi\} \equiv \mathrm{e}^{\mathscr{A}\{\tilde{\phi},\phi\}} \tag{5}$$

will play an important role in our treatment.

In our case the action reads

$$\mathscr{A}\{\tilde{\phi}, \phi\} = \int \mathrm{d}t \int \mathrm{d}^d x \left(\sum_{j=1}^{n} [-\Gamma_0 \tilde{\phi}_j^2 + \mathrm{i}\tilde{\phi}_j(\dot{\phi}_j - \Gamma_0 a_0 \Delta\phi_j)] + \varphi\Gamma_0 t(\phi^2) \right), \tag{6}$$

with the notation

$$t(\phi^2) \equiv \mathrm{d}U(\phi^2)/\mathrm{d}\phi^2, \tag{7}$$

$$\varphi \equiv \mathrm{i}\sum_{j=1}^{n} \tilde{\phi}_j\phi_j + \frac{n}{2}K_\Lambda, \qquad K_\Lambda \equiv \int_0^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d} = \frac{K_d}{d}\Lambda^d. \tag{8}$$

The contribution $(n/2)K_\Lambda\Gamma_0 t(\phi^2)$ comes from the logarithm of the functional Jacobian, whose other term $K_\Lambda\Gamma_0\phi^2\, \mathrm{d}t(\phi^2)/\mathrm{d}\phi^2$ is neglected since for large n it is of the order 1.

3. The renormalisation group transformation

The renormalisation group is defined as a transformation on W. The weight $W\{\tilde{\phi}, \phi\}$ is to be integrated over the large-momentum variables, i.e. over the Fourier components $\tilde{\phi}_{j,k,\omega}$ and $\phi_{j,k,\omega}$ with k values between Λ/b and $\Lambda (b > 1)$ and arbitrary ω.

As a first step we separate the fields into two parts involving small- and large-wavenumber components respectively:

$$\tilde{\phi}_j(x, t) \to \tilde{\phi}_j(x, t) + \tilde{\phi}'_j(x, t) = \tilde{\phi}_j(x, t) + L^{-d/2} \sum{}' \tilde{\phi}_{j,k,\omega}\, \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tag{9}$$

and similarly for $\phi_j(x, t)$. Here L^d is the volume of the system and

$$\sum{}' \equiv \sum_{\Lambda/b < |k| < \Lambda, \omega}. \tag{10}$$

Now proceeding in the same spirit as Ma (1973), in the statics of the model we approximate

$$\frac{1}{2}\sum_{j=1}^{n}(\phi_j+\phi_j')^2 \approx \frac{1}{2}\sum_{j=1}^{n}\phi_j^2(x,t)+\rho, \tag{11}$$

where

$$\rho = L^{-d}\sum{}' N_{k,\omega}, \qquad N_{k,\omega} \equiv \frac{1}{2}\sum_{j=1}^{n}|\phi_{j,k,\omega}|^2, \tag{12}$$

and similarly

$$\sum_{j=1}^{n}(\tilde{\phi}_j+\tilde{\phi}_j')(\phi_j+\phi_j') \approx \sum_{j=1}^{n}\tilde{\phi}_j(x,t)\phi_j(x,t)+L^{-d}\sum_{j=1}^{n}\sum{}'\tilde{\phi}_{j,k,\omega}\cdot\phi_{j,-k,-\omega}. \tag{13}$$

These approximations amount to neglecting $\phi_j\phi_j'$, $\tilde{\phi}_j\phi_j'$ and $\tilde{\phi}_j'\phi_j$ and also those terms in $\phi_j'^2$ and $\tilde{\phi}_j'\phi_j'$, when expressed in terms of Fourier transforms of ϕ_j' and $\tilde{\phi}_j'$, which are non-diagonal in k and ω. The former makes an error of the order $1/n$, which is negligible in the large-n limit. The latter means that we do not treat the wavenumber and frequency dependences of the couplings. We will return to a discussion of this question later. After substituting equations (9), (11) and (13) into (6) we integrate W over $\tilde{\phi}_{j,k,\omega}(\Lambda/b<|k|<\Lambda)$, which is straightforward as $\mathscr{A}$ is quadratic in $\tilde{\phi}$. This yields a new action

$$\hat{\mathscr{A}}\{\tilde{\phi},\phi\} = \int \mathrm{d}t \int \mathrm{d}^d x\bigg(\sum_{j=1}^{n}[-\Gamma_0\tilde{\phi}_j^2+\mathrm{i}\tilde{\phi}_j(\dot{\phi}_j-\Gamma_0 a_0\Delta\phi_j)]$$

$$+\varphi\Gamma_0 t(\phi^2+\rho)-\frac{1}{2\Gamma_0}L^{-d}\sum{}'[\omega^2+\Gamma_0^2(a_0k^2+t(\phi^2+\rho))^2]N_{k,\omega}\bigg). \tag{14}$$

Now the integration of $\hat{\mathscr{A}}$ over $\phi_{j,k,\omega}(\Lambda/b<|k|<\Lambda)$ has to be performed. $\hat{\mathscr{A}}$ depends on these variables only through $N_{k,\omega}$, which will be introduced as a new integration variable. Similarly to the situation in statics (Ma 1973), the integrand turns out to have a very sharp maximum, making the use of the saddle point method possible. Doing so and introducing new scales

$$x \to bx, \qquad t \to b^z t, \tag{15}$$

$$\phi(x,t) \to b^{1-\eta/2-d/2}\phi(x,t), \qquad \tilde{\phi}(x,t) \to b^{-1+\eta/2-d/2}\tilde{\phi}(x,t), \tag{16}$$

where in the usual notation η and z are the static correlation function exponent and the dynamic critical exponent respectively, we obtain a new action $\mathscr{A}\{\tilde{\phi},\phi\}$ in a more general form than that given by equation (6):

$$\mathscr{A}\{\tilde{\phi},\phi\} = \int \mathrm{d}t \int \mathrm{d}^d x\bigg(\sum_{j=1}^{n}[-\Gamma\tilde{\phi}_j^2+\mathrm{i}\tilde{\phi}_j(\dot{\phi}_j-\Gamma a\Delta\phi_j)]+Y(\phi^2,\varphi)\bigg). \tag{17}$$

Here $Y(\phi^2,\varphi)$ is a function of two variables which cannot be factorised any more. $Y(\phi^2,\varphi=0)$ turns out to be a constant which will not be regarded as a parameter in $\mathscr{A}$, so the derivative

$$y(\phi^2,\varphi) \equiv \partial Y(\phi^2,\varphi)/\partial\varphi \tag{18}$$

specifies all the parameters besides a and Γ. We obtain

$$a = a_0 b^{-\eta}, \qquad \Gamma = \Gamma_0 b^{z-2+\eta}. \tag{19}$$

It follows that a necessary condition for the existence of a finite fixed point is

$$\eta = 0, \qquad z = 2, \tag{20}$$

which are well-known results for $n = \infty$. So equation (19) does not involve the transformation of a_0 or Γ_0, and we can choose $a_0 = 1, \Gamma_0 = 1$.

The steps leading to the expression for $y(\phi^2, \varphi)$ are given in the appendix. We arrive at

$$y(\phi^2, \varphi) = b^2 t(b^{2-d}\phi^2 + \bar{\rho}), \tag{21}$$

where

$$\bar{\rho} \equiv \bar{\rho}(\phi^2, \varphi) = \frac{n}{2} b^{2-d} \int_{\Lambda}^{\Lambda b} \frac{\mathrm{d}^d q/(2\pi)^d}{[(q^2 + y(\phi^2, \varphi))^2 - X(\phi^2, \varphi)]^{1/2}} \tag{22}$$

and

$$X(\phi^2, \varphi) = b^{4-d} 2v(b^{2-d}\phi^2 + \bar{\rho}) \times \left[\varphi - \frac{n}{2} \int_{\Lambda}^{\Lambda b} \left(\frac{q^2 + y(\phi^2, \varphi)}{[(q^2 + y(\phi^2, \varphi))^2 - X(\phi^2, \varphi)]^{1/2}} - 1\right) \frac{\mathrm{d}^d q}{(2\pi)^d}\right], \tag{23}$$

with $v(\xi) = \mathrm{d}t(\xi)/\mathrm{d}\xi$. Equations (21)–(23) specifying y form a complicated self-consistent set of equations. In what follows it will be assumed that v is non-zero, since the opposite case would mean that we started from the Gaussian model.

4. Fixed points and stability

Let us now turn to the determination of the fixed point expression y^* of the function y generated by the limit $b \to \infty$. Only the case $d > 2$ will be discussed. As a first step we examine equation (23) in the limit $b \to \infty$. It is obvious that in the case $d < 4$ a finite fixed point can be achieved only if we have

$$\varphi = \frac{n}{2} \int_{\Lambda}^{\infty} \left(\frac{q^2 + y^*(\phi^2, \varphi)}{[(q^2 + y^*(\phi^2, \varphi))^2 - X^*(\phi^2, \varphi)]^{1/2}} - 1\right) \frac{\mathrm{d}^d q}{(2\pi)^d} \tag{24}$$

at the fixed point.

Moreover for dimensions $2 < d < 4$ it can be easily seen that, through steps analogous to those carried out by Ma (1973) in statics, one obtains from equations (21) and (22) in the limit $b \to \infty$ at $T = T_c$

$$\phi^2 = \frac{n}{2} \frac{K_d \Lambda^{d-2}}{d-2} - \frac{n}{2} \int_{\Lambda}^{\infty} \left(\frac{1}{[(q^2 + y^*(\phi^2, \varphi))^2 - X^*(\phi^2, \varphi)]^{1/2}} - \frac{1}{q^2}\right) \frac{\mathrm{d}^d q}{(2\pi)^d}. \tag{25}$$

The coupled integral equations (24) and (25) give y^* as a function of ϕ^2 and φ at the non-trivial fixed point. For $d > 4$ a Gaussian fixed point becomes stable with $X^* \equiv 0, y^* \equiv 0$.

It can be seen that, for $\varphi \to 0, X^*(\phi^2, \varphi) \to 0$ and equation (25) reduces to the one derived by Ma (1973, 1974a) in statics for his $t^*(\phi^2)$ when $2 < d < 4$. For $\varphi \neq 0$ the integral equations (24) and (25) can be solved in the form of a power series:

$$y^*(\phi^2, \varphi) = u_2^* + u_4^* \phi^2 + \bar{u}_4^* \varphi + \tfrac{1}{2} u_6^* (\phi^2)^2 + \bar{u}_6^* \phi^2 \varphi + \tfrac{1}{2} \bar{\bar{u}}_6^* \varphi^2 + \ldots \tag{26}$$

Here $u_2^*, u_4^*, u_6^*, \ldots$ are static parameters already fixed by static RG (Ma 1973). We recall the results in the most usual notation, $u_2^* = r^*$ and $u_4^* = 2u^*$:

$$1 = -r^*(d-2)\Lambda^{2-d}\int_\Lambda^\infty \frac{q^{d-3}\,\mathrm{d}q}{q^2+r^*},$$

$$u^* = \left(n\int_\Lambda^\infty \frac{1}{(q^2+r^*)^2}\frac{\mathrm{d}^dq}{(2\pi)^d}\right)^{-1},$$

$$u_6^* = 8n(u^*)^3\int_\Lambda^\infty \frac{1}{(q^2+r^*)^3}\frac{\mathrm{d}^dq}{(2\pi)^d}.$$

The dynamic quantities are obtained as

$$\bar{u}_4^* = 2nu^{*2}\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^3}, \tag{27}$$

$$\bar{u}_6^* = 16n^2u^{*4}\left(\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^3}\right)^2 - 12nu^{*3}\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^4}, \tag{28}$$

$$\bar{\bar{u}}_6^* = 24n^3u^{*5}\left(\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^3}\right)^3 - 36n^2(u^*)^4\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^4}\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^3}$$
$$+12nu^{*3}\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{(q^2+r^*)^5}. \tag{29}$$

Besides determining the fixed point in the parameter space, it is of primary interest how the fixed point is approached. This can be easily evaluated from equations (21)–(23). In leading order for large b it is found that

$$\Delta y' = b^{d-4}\frac{\left(\dfrac{\Lambda^2}{(d-2)u_\mathrm{c}} - \dfrac{1}{4-d}\right)K_d\Lambda^{d-4}}{X^*\left(\displaystyle\int_\Lambda^\infty \frac{\mathrm{d}^dq/(2\pi)^d}{[(q^2+y^*)^2-X^*]^{3/2}}\right)^2 - \left(\displaystyle\int_\Lambda^\infty \frac{(q^2+y^*)\mathrm{d}^dq/(2\pi)^d}{[(q^2+y^*)^2-X^*]^{3/2}}\right)^2}$$
$$\times\int_\Lambda^\infty \frac{y^*(q^2+y^*)+X^*/2}{[(q^2+y^*)^2-X^*]^{3/2}}\frac{\mathrm{d}^dq}{(2\pi)^d} \tag{30}$$

provided our starting point in the parameter space lies on the critical surface ($T = T_\mathrm{c}$). The condition for this is (Ma 1973) that

$$t_1 \equiv t(N_\mathrm{c}) = 0, \qquad N_\mathrm{c} = (n/2)K_d\Lambda^{d-2}/(d-2).$$

Furthermore the quantity u_c in equation (30) is a property of the critical surface:

$$u_\mathrm{c} \equiv N_\mathrm{c}\,\mathrm{d}t(\xi)/\mathrm{d}\xi\Big|_{\xi=N_\mathrm{c}}.$$

In the static case obtained formally by setting $X^* = 0$, $\varphi = 0$, and writing $y^*(\phi^2, 0) = t^*(\phi^2)$, equation (30) becomes

$$\Delta t' = b^{d-4}t^*\left(-\frac{\Lambda^2}{(d-2)u_\mathrm{c}} + \frac{1}{4-d}\right)\Lambda^{d-4}\Big/\int_\Lambda^\infty \frac{q^{d-1}\,\mathrm{d}q}{(q^2+t^*)^2},$$

a result agreeing with that of Ma (1974a).

Equation (30) makes it possible to estimate the size of the critical region at T_c by the condition that the deviations of the irrelevant parameters from their fixed point values be negligible. In dynamics one can estimate the size of the critical region in ω by choosing $b \simeq \Lambda/\omega^{1/2}$.

It is of particular interest that the size of the critical region when dynamic properties are also taken into consideration is influenced by the fixed point values of generated higher-order dynamic parameters.

t_1 as defined above is a relevant variable which measures the distance from the critical surface. At a temperature T, very close to T_c, $t_1 \simeq |T - T_c|$ and

$$\Delta y' \simeq t_1 b^{d-2},$$

which gives $\nu = 1/(d-2)$, a result known from statics (see Ma 1973).

Let us now turn to a short discussion of the space and time, or equivalently k and ω, dependences of the parameters which were not treated in the calculation above. The coefficients of the powers of k and ω define another set of parameters, i.e. another direction in the parameter space. If we want to study this question we have to rely on a perturbation expansion of W and treat each coupling separately. In this way we derived an explicit expression for the quartic coupling $u^*(k, \omega)$. It is a regular function which for small k and ω can be expanded as

$$u^*(k, \omega) = u^* + A^*k^2 + B^*\mathrm{i}\omega + \ldots, \tag{31}$$

where

$$A^* = nu^{*2} \int_\Lambda^\infty \frac{K_d q^{d-1}}{(q^2 + r^*)^3}\left(1 - \frac{4q^2/d}{q^2 + r^*}\right) \mathrm{d}q, \tag{32}$$

$$B^* = -nu^{*2} \int_\Lambda^\infty \frac{K_d q^{d-1}\,\mathrm{d}q}{2(q^2 + r^*)^3} \tag{33}$$

and u^* is the same as in equation (26).

It can easily be seen that parameters such as A and B are also irrelevant ones which approach their fixed point values as fast as b^{d-4}:

$$\Delta B' = b^{d-4}\left(\frac{\Lambda^2}{(d-2)u_c} - \frac{1}{4-d}\right) K_d \Lambda^{d-4} \frac{1}{2}\left[2n^2u^{*3} \int_\Lambda^\infty \frac{\mathrm{d}^d q/(2\pi)^d}{(q^2 + r^*)^3}\right.$$
$$\left. + 4n^3u^{*4}r^*\left(\int_\Lambda^\infty \frac{\mathrm{d}^d q/(2\pi)^d}{(q^2 + r^*)^3}\right)^2 - 3n^2u^{*3}r^* \int_\Lambda^\infty \frac{\mathrm{d}^d q/(2\pi)^d}{(q^2 + r^*)^4}\right].$$

5. Discussion

In concluding we first comment on the important point that an infinite number of new dynamic parameters $\bar{u}_4$, $\bar{u}_6$, $\bar{\bar{u}}_6$, etc are generated by the DRG, a property which is general, i.e. it holds also when n is not large, for dimensions $2 < d < 4$. These parameters can be given the following interpretation. If the RG transformation is performed directly on the equation of motion, one obtains a new equation of motion in which not only the Langevin noise but also the vertices are random variables. All the cumulants of these quantities are to be considered as the elements of the parameter space (Halperin *et al* 1976, Sasvári and Szépfalusy 1977). The quantities $\bar{u}_4^*$, $\bar{u}_6^*$, $\bar{\bar{u}}_6^*, \ldots$ appearing in the present formulation are strongly related to the fixed point values of the aforementioned cumulants at zero wavenumber and zero frequency.

From the point of view of the appearance of new parameters, the $n \to \infty$ limit is more complicated than the small ϵ limit. If one calculates to $O(\epsilon)$, the parameters $\bar{u}_4^*, \bar{u}_6^*, \bar{\bar{u}}_6^*, \dots$ as well as parameters such as A^* and B^* are negligible as they are of order ϵ^2 or higher. This can be easily seen from equations (27)–(29), (31)–(33) if we take into account the well-known result that u^* is of the order ϵ. In the large n limit, however, the higher-order couplings give contributions to the action which are of the same order in n. The situation is similar as in statics, where $u_{2m} = O(1/n^{m-1})$, and then all contributions to the free energy are equally large, $O(n)$. As for dynamics, it can be seen that $\bar{u}_{2m}$, $\bar{\bar{u}}_{2m}$, etc are of the same order as u_{2m}. As the higher-order couplings are irrelevant parameters, their importance lies in influencing the extent of the critical region in which dynamic scaling is valid.

We have used as the starting point in our investigation an action which does not contain these higher-order dynamic parameters but only the static ones. This can be justified by applying the general considerations first worked out for statics by Wilson and Kogut (1974) and Wilson (1975) and generalised to dynamics by Halperin *et al* (1976). The parameters specifying the higher-order vertices and the k- and ω-dependent parts of u_4 and of the other couplings are 'fast transients' which relax rapidly to quasi-stationary values that are independent of the choice of the original values of these parameters and are determined entirely by the slow parameters r and u. These considerations also mean that for our starting Hamiltonian (equation (1)) a Ginzburg–Landau form could have been chosen. Concerning the static properties of the model, this question has been investigated by Ma (1974a). His result demonstrates that the higher-order bare static couplings play a role only by specifying the critical surface in the parameter space—a property which is obviously not influenced by any dynamic parameter.

Our aim here has not been to calculate the dynamic critical exponent in the $1/n = 0$ case, which is well known and trivial, but to investigate the details of the Wilson-type DRG in the large-n limit. The dynamical critical exponent has actually been determined up to the next order by Halperin *et al* (1972) and by De Dominicis *et al* (1975). The latter authors did not work in the framework of the Wilson-type RG, but applied field theoretic methods based on the Callan–Symanzik equation and carried out the calculations in the large-cut-off limit, which considerably simplifies the parameter space. In deriving the dynamic critical exponent to the order $1/n$, Halperin *et al* (1972) used the Wilson-type matching condition, in contrast to their calculation for small ϵ, for which the full analysis of the RG has been carried out as well (Halperin *et al* 1976). In this context our results provide an RG background for their large-n expansion.

Appendix

The renormalisation group transfo mation is defined by

$$R_b = R_b^s R_b^i, \tag{A1}$$

where

$$e^{R_b^i \mathscr{A}\{\tilde{\phi},\phi\}} = \int \prod_{j,\Lambda/b<k<\Lambda,\omega} d\tilde{\phi}_{j,k,\omega}\, d\phi_{j,k,\omega}\, e^{\mathscr{A}\{\tilde{\phi},\phi\}} \tag{A2}$$

and R_b^s stands for the change of scales according to equations (15) and (16).

After carrying out the integrations over the $\tilde{\phi}$ fields, the right-hand side of equation (A2) can be written as

$$\int \prod_{j,\Lambda/b<k<\Lambda,\omega} \mathrm{d}\phi_{j,k,\omega}\, \mathrm{e}^{\mathscr{A}\{\tilde{\phi},\phi\}} \propto \int \prod_{\Lambda/b<k<\Lambda,\omega} \mathrm{d}N_{k,\omega}\, N_{k,\omega}^{n-1}\, \mathrm{e}^{\hat{\mathscr{A}}\{\tilde{\phi},\phi\}}, \tag{A3}$$

where $\hat{\mathscr{A}}$ and $N_{k,\omega}$ are given by equations (14) and (12) respectively. In the large-n limit the last integral can be approximated by the maximum of its integrand, yielding

$$R_b^i\mathscr{A}\{\tilde{\phi},\phi\} = \int \mathrm{d}^d x \int \mathrm{d}t \left(\sum_{j=1}^{n} [-\Gamma_0\tilde{\phi}_j^2 + \mathrm{i}\tilde{\phi}_j(\dot{\phi}_j - \Gamma_0 a_0 \Delta\phi_j)] + w(\phi^2, \varphi, \bar{N}_{k,\omega}) \right). \tag{A4}$$

Here

$$w(\phi^2, \varphi, N_{k,\omega}) = \varphi\Gamma_0 t(\phi^2+\rho) + \frac{n}{2} L^{-d} \sum{}' \ln N_{k,\omega}$$
$$- \frac{1}{2\Gamma_0} L^{-d} \sum{}' \{\omega^2 + \Gamma_0^2[a_0 k^2 + t(\phi^2+\rho)]^2\} N_{k,\omega} \tag{A5}$$

and $\bar{N}_{k,\omega}$ is obtained through the condition

$$\partial w/\partial N_{k,\omega}|_{N_{k,\omega}=\bar{N}_{k,\omega}} = 0 \tag{A6}$$

as

$$\bar{N}_{k,\omega} = \frac{n\Gamma_0}{\omega^2 + \Gamma_0^2[a_0k^2 + t(\phi^2+\bar{\rho})]^2 - 2\Gamma_0^2(\mathrm{d}t/\mathrm{d}\phi^2)v(\phi^2+\bar{\rho})\{\varphi - L^{-d}\sum{}'[a_0k^2 + t(\phi^2+\bar{\rho})]\bar{N}_{k,\omega}\}}. \tag{A7}$$

By comparing equations (A4) and (6) we can write symbolically

$$w = R_b^i Y_0, \tag{A8}$$

where

$$Y_0 = \Gamma_0 \varphi t(\phi^2). \tag{A9}$$

The subsequent operation R_b^s yields an action of the form of equation (17). The transformations for a_0 and Γ_0 given by equation (19) can be read off simply. Furthermore it is found that

$$Y(\phi^2, \varphi) = b^{d+z} w(\phi^2, \varphi, \bar{N}_{k,\omega}) \Big|_{\substack{\phi(x,t)\to b^{1-\eta/2-d/2}\phi(x,t) \\ \tilde{\phi}(x,t)\to b^{-1+\eta/2-d/2}\tilde{\phi}(x,t).}} \tag{A10}$$

Finally, taking into account equations (A5), (A6) and the definition equation (18) we arrive at equation (21). The expression for $\bar{\rho}$ given by equation (22) can be obtained by substituting equation (A7) into equation (12) and integrating over ω.

Note added in proof. Concerning the approach to the fixed point as given for large b by equation (30), we note that more generally for arbitrary b the non-linear scaling fields can be given as $g_{\alpha\beta} = a_{\alpha\beta} - a^*_{\alpha\beta}$ $\alpha = 1, 2, \ldots,$ $\beta = 0, 1, 2; \ldots$ where the parameters $a_{\alpha\beta}$ are the Taylor coefficients of the Legendre transform of $Y(\phi^2, \varphi)$. The corresponding exponents are $y_{\alpha\beta} = d + 2 - 4\alpha - 2\beta$.

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